Collection of Wake-Field Formulae

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We assume a Gaussian longitudinal distribution, and generally use cgs units. Formulae can be converted to SI units by multiplying or dividing with $Z_0 c/(4\pi)$, where Z_0 is the vacuum impedance and c the speed of light.

1 General Transverse Wake

The transverse kick factor κ_t is defined by

$$\Delta y' = \frac{N r_e}{\gamma} \kappa_t \, y \tag{1}$$

where N is the bunch population, r_e the classical electron radius, y the centroid displacement of the bunch, and $\Delta y'$ the centroid deflection.

In terms of jitter amplification this can also be written as

$$\frac{\Delta y'}{\sigma_{y'}} = \frac{N \beta r_e}{\gamma} \kappa_t \, \frac{y}{\sigma_y} \equiv \mu_t \frac{y}{\sigma_y} \tag{2}$$

where β is the beta function at the wake-field source, σ_y, the rms beam size and $\sigma_{y'}$ the rms divergence. We call the parameter μ_t the jitter amplification factor.

The kick factor is related to the wake function $w(s)$ by

$$\kappa_t = \int f(z) f(z') w_t(z - z') dz \, dz' \tag{3}$$

where $f(z)$ denotes the normalized longitudinal distribution, and to the impedance via [2]

$$\kappa_t = -\frac{1}{\pi} \int_0^\infty d\omega \, |\tilde{F}(\omega)|^2 \mathrm{Im} Z_t(\omega) = \int_0^\infty d\omega F \left(\frac{\omega}{c} \sigma_z \right) \mathrm{Re} Z_t(\omega) \tag{4}$$

where, for a Gaussian distribution,

$$F(x) = -\frac{i}{\pi} \exp(-x^2) \, \mathrm{erf}(ix), \tag{5}$$

and, for large x, $F(x) \approx \pi^{-3/2} x^{-1} \left(1 + \frac{1}{2x^2}\right))$.

2 Geometric Wake for Shallow Circular Taper

The longitudinal impedance of a shallow circular taper, calculated by Yokoya [1], is

$$Z_t(\omega) = \frac{i \omega Z_0}{4 \pi c} \int_\infty^\infty dz (b')^2, \tag{6}$$

and the transverse impedance [1]

$$Z_t = -\frac{i Z_0}{2\pi} \int_\infty^\infty dz \left(\frac{b'}{b} \right)^2, \tag{7}$$

where $b(z)$ denotes the beam pipe radius as a function of z and $Z_0 = 120\pi \, \Omega$. Stupakov [2] has shown that this formula is valid for frequencies

$$\omega b^2 / l \ll c. \tag{8}$$

From Eq. (7) one obtains the kick factor

$$\kappa_t = \frac{\theta}{\sqrt{\pi}\sigma_z}\left(\frac{1}{g} - \frac{1}{b}\right) \tag{9}$$

where g is the smaller beam-pipe radius, b the larger radius, l the length of the taper, $\theta = (b-g)/l$ the taper angle, and σ_z the bunch length.

Approximating $\theta \approx l/b$, $\omega \approx c/\sigma_z$ we can rewrite the applicability condition of Eq. (8) as

$$\sigma_z \gg b\theta \tag{10}$$

with b the beam-pipe radius and θ the taper angle. The rms kick is about 0.4 times the centroid kick.

Example: $\sigma_z = 30\mu m$, $b = 5$ *mm, and formula is applicable if* $\theta \gg 6$ *mrad.*
Take $\theta = 10$ *mrad,* $\gamma = 2.9 \times 10^6$, $N = 4 \times 10^9$, $\beta = 10^5$ *m,* $b = 5$ *mm,* $g = 1$ *mm (this corresponds to about 40* σ_y*!), then* $r_e N\beta/\gamma \approx 4 \times 10^{-7}$ *m*^{2}, $\kappa_t = 1.5 \times 10^5$ *m*$^{-2}$*, and* $\mu_t = 0.06$*. Note that this should be multiplied by a factor of two to account for the contributions from tapers on either side.*

3 Geometric Wake for Tapered Parallel-Plate Step

For a rectangular taper, Stupakov [3, 4] has deduced the following result:

$$\Delta y'(z) = \frac{N r_e}{\sqrt{2\pi}\gamma\sigma_z} e^{-z^2/(2\sigma_z^2)} \left[(2\pi w I_2 - 2I_1)y_0 + 2I_1 y\right] \tag{11}$$

where y_0 is the centroid position, y the coordinate of the test particle, and w the half width of the collimator.

The integrals I_1 and I_2 are

$$I_1 = \int \frac{b'^2}{b^2}\, ds = \theta\left(\frac{1}{g} - \frac{1}{b}\right) \tag{12}$$

and

$$I_2 = \int \frac{b'^2}{b^3}\, ds = \frac{\theta}{2}\left(\frac{1}{g^2} - \frac{1}{b^2}\right) \tag{13}$$

where b is the vertical half gap at the entrance and g that at the end of the taper. With $y = y_0$ this translates into a kick factor

$$\kappa_t = \frac{0.282\pi w\theta}{\sigma_z}\left(\frac{1}{g^2} - \frac{1}{b^2}\right). \tag{14}$$

As in the circular case, the rms kick is about 0.4 times the centroid kick.

Formula (14) is valid if

$$\frac{\omega}{g} < \frac{b-g}{l}. \tag{15}$$

Preliminary experimental results indicate that the wake field may be smaller if this condition is violated [5].

4 Geometric Wake for Untapered Circular Step

If there is no taper, for short bunches, $\sigma_Z \ll g$, the transverse kick factor reads [6, 7]

$$\kappa_t = \left(\frac{1}{g^2} - \frac{1}{b^2}\right) \tag{16}$$

and the rms kick is about $1/\sqrt{3}$ times this value.

Example: for $\sigma_z = 30\mu m$, $b = 5$ *mm,* $g = 1$ *mm, we find* $\kappa_t \approx 10^6$ *m*$^{-2}$*, and* $\mu_t = 0.37$*. Note again that this should be multiplied by a factor of two to account for the contributions from tapers on either side.*

The complete nonlinear kick, for any offset y, is [8, 7]:

$$\Delta y'(z) = \frac{4Nr_e}{\gamma}\left(\frac{1}{g^2 - y^2} - \frac{1}{b^2 - y^2}\right) y \int_{-\infty}^{z} g(z) \tag{17}$$

where

$$g(z) = \frac{\exp\left(\frac{-z^2}{2\sigma_z^2}\right)}{\sqrt{2\pi}\sigma_z} \tag{18}$$

is the normalized beam distribution.

5 Resistive Wake between Two Parallel Plates

Wake field near axis for beam passing through two parallel plates of length L with half gap b [9, 10]:

$$\Delta y' = \frac{\pi r_e N L}{4b^2\gamma}\left(\frac{c}{\sigma\sigma_z}\right) f(s/\sigma_z)\left(\frac{y}{b}\right) \tag{19}$$

with σ the conductivity, and

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-(y-x)^2/2}}{\sqrt{y}}\, dy \tag{20}$$

In a round collimator the result is $8/\pi^2$ smaller. Note that according to Ref. [9] $< f(s) >= 0.78$ for a Gaussian charge distribution. We calculated $< f(s) >= 0.72$, numerically integrating with mathematica. We also computed the rms kick: $< f(s)^2 - < f(s) >^2 >^{1/2} = 0.29$.

The complete nonlinear wake near the wall is obtained by replacing in Eq. (19) the factor y/a by [11, 9]

$$\frac{1}{\pi}\left(\frac{\pi y/b + \sin \pi y/b}{1 + \cos \pi y/b}\right) \tag{21}$$

For $y \to a$ this diverges as $1/(a-y)^2$.

Example: $\sigma = 3.2 \times 10^{17}\ s^{-1}$ *for Al,* $\sigma = 5.4 \times 10^{17}\ s^{-1}$ *for Cu, and* $\sigma = 1.3 \times 10^{16}\ s^{-1}$ *for stainless steel. Assume copper.* $L = 14$ *cm (10 radiation lengths),* $b = 1$ *mm, and* $\beta = 10^5$. *This yields* $\mu_t = 0.13$.

6 Optimum Taper

The optimum taper angle is obtained when the resistive and geometric kicks are equal. For circular geometry and $g \ll b$ it is [4]

$$\theta_{opt} \approx 0.6\left(\frac{c\sigma_z}{\sigma g^2}\right)^{1/4} \tag{22}$$

Example: Cu with $\sigma = 5.4 \times 10^{17}\ s^{-1}$, $g = 1$ *mm, and* $\theta \approx 7$ *mrad.*

7 General Longitudinal Wake

The longitudinal wake function is related to the real part of the impedance via

$$W_0(z) = \frac{2}{\pi}\int_0^\infty \mathrm{Re}Z^{\|}(\omega)\cos\frac{\omega z}{c} \tag{23}$$

The total energy loss is

$$\Delta E = -\int_{-\infty}^{\infty} dz' \rho(z') \int_{z'}^{\infty} dz \rho(z) W_0(z-z') = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\tilde{\omega}|^2 \mathrm{Re}Z_0(\omega) \tag{24}$$

where $\rho(z)$ is the charge distribution, and positive z refers to the tail of the bunch.

8 Longitudinal Resistive-Wall Wake

The resistive-wall wake in a circular beam pipe is characterized by the length parameter

$$s_0 = \left(\frac{cb^2}{2\pi\sigma}\right)^{1/3} \quad \text{(cgs units)} \tag{25}$$

where b is the chamber radius and σ the conductivity.

Example: Cu and $b = 5$ *mm yields* $s_0 \approx 13\ \mu$ *m.*

When the bunch length is long compared with s_0 the field along a Gaussian bunch can be written [12]

$$E_z(s/\sigma_z) = \frac{Ne}{4b^2}\left(\frac{s_0}{\sigma_z}\right)^{3/2} f\left(\frac{s}{\sigma_z}\right) \tag{26}$$

with

$$f(u) = |u|^{3/2} e^{-u^2/4} \left[I_{1/4}\left(\frac{u^2}{4}\right) - I_{-3/4}\left(\frac{u^2}{4}\right) \mp I_{-1/4}\left(\frac{u^2}{4}\right) \pm I_{3/4}\left(\frac{u^2}{4}\right) \right] \tag{27}$$

Here I are the modified Bessel functions and upper signs are evaluated for $u < 0$ while the lower signs are taken for $u > 0$. The function $f(u)$ is depicted in Fig. 1.

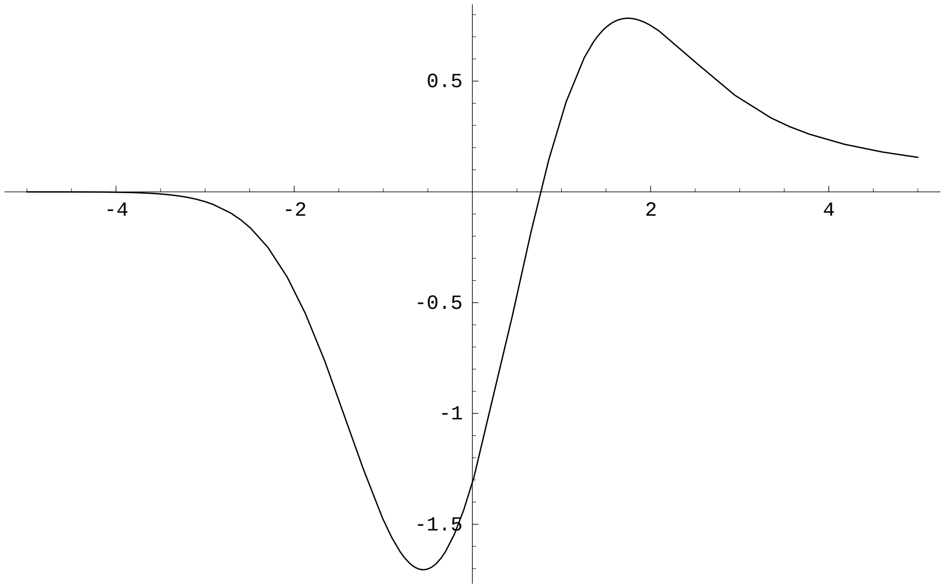

Figure 1: Function $f(u)$ versus u. The bunch head is on the left.

From this one obtains the average energy loss per unit length [13]

$$\frac{dE}{ds} \approx 0.195 \frac{Ne^2}{b^2} \left(\frac{s_0}{\sigma_z}\right)^{3/2} \quad \text{(cgs units)} \tag{28}$$

and the rms energy spread

$$\frac{dE}{ds} \approx 0.205 \frac{Ne^2}{b^2} \left(\frac{s_0}{\sigma_z}\right)^{3/2} \quad \text{(cgs units)} \tag{29}$$

For Copper this can be expressed as

$$(\Delta\delta)_{\rm rms} \approx 0.88 \, \frac{N[10^{10}]}{E[\text{eV}] b[\text{cm}] (\sigma_z[\text{mm}])^{3/2}} \Delta s[\text{cm}] \tag{30}$$

For bunch lengths approaching the characteristic length s_0 the wake changes the shape [14, 15]. However, Eqs. (28) and (29) are still good approximations up to bunch lengths equal to s_0. For even shorter bunches they overestimate the energy spread.

Assuming a constant (in ω) conductivity the short-range wake is [14]

$$E_z(\tilde{s}) = -\frac{Ne}{b^2} \left(\frac{16}{3} \exp(-\tilde{s}) \, \cos(\sqrt{3}\tilde{s}) - \frac{16\sqrt{2}}{\pi} \int_0^\infty \frac{dx \, x^2 \exp(-x^2 \tilde{s})}{x^6 + 8} \right) \tag{31}$$

with $\tilde{s} = s/s_0$. For tracking simulations, this can be approximated as [13]

$$E_z(\tilde{s}) = -\frac{Ne}{b^2} \left(\frac{16}{3} \exp(-\tilde{s}) \, \cos(\sqrt{3}\tilde{s}) - \frac{1}{\sqrt{2\pi}} \frac{1}{(a^\alpha + \tilde{s}^{\alpha 3/2})^{1/\alpha}} \right) \tag{32}$$

where $a = 3/(4\sqrt{(2\pi)})$ to give the right value at $\tilde{s} = 0$, and $\alpha \approx 0.9$.

9 Longitudinal Geometric Wake

For an untapered circular collimator (or step-out) and bunch lengths much smaller than the transverse aperture, the impedance is [16, 17, 18]

$$Z^{\|}_{\rm step} = \frac{Z_0}{\pi} \ln \frac{b}{g}, \tag{33}$$

and the wake function is approximately a δ function [13],

$$W_{0,\rm step}(z) = \frac{2cZ_0}{\pi} \ln \frac{b}{g}\, \delta(z). \tag{34}$$

Then the average change in energy reads

$$(\Delta\delta)_{\rm ave} = \frac{2Nr_E}{\sqrt{\pi}\gamma\sigma_z} \ln \frac{b}{g} \approx 1.13 \times \frac{Nr_e}{\gamma\sigma_z} \ln \frac{b}{g} \tag{35}$$

and the induced rms energy spread

$$(\Delta\delta)_{\rm rms} = \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)^{1/2} \frac{4Nr_E}{\sqrt{2\pi}\gamma\sigma_z} \ln \frac{b}{g} \approx 0.444 \times \frac{Nr_e}{\gamma\sigma_z} \ln \frac{b}{g}. \tag{36}$$

Similar to the transverse case, the energy loss can be reduced by a taper. For a double-sided tapered transition in the high-frequency limit, the longitudinal loss factor is given by [17]

$$(\Delta\delta)^{\rm taper}_{\rm ave} = \frac{2Nr_e}{\sqrt{\pi}\gamma\sigma_z}(1-\tilde{\eta}_1)\ \ln \frac{b}{g} \approx 1.13 \times \frac{Nr_e}{\gamma\sigma_z}(1-\tilde{\eta}_1) \ln \frac{b}{g} \tag{37}$$

where $\tilde{\eta}_1 = \min(1.0, \eta_1)$ with

$$\eta_1 = \frac{l\sigma_z}{(b-g)^2} \approx \frac{\sigma_z}{\alpha(b-g)} \tag{38}$$

wher el is the length of the taper and $\alpha = \arctan((b-g)/l) \approx (b-g)/l$ the taper angle with respect to the beam direction. The rms energy variation is similarly

$$(\Delta\delta)^{\rm taper}_{\rm rms} \approx 0.444 \times \frac{Nr_e}{\gamma\sigma_z}(1-\tilde{\eta}_1) \ln \frac{b}{g} \tag{39}$$

If the taper is sufficiently tapered $\tilde{\eta}_1 = 1.0$ and there is no energy loss.

In the case of many closely spaced transitions, there can be an interference between the electromagnetic waves generated at different transitions. For the special situations of a peridoic array of M cavities and $M \gg \omega g^2/(cL)$ (with g the iris radius) and L the longitudinal period, Heifets and Kheifets [17] have shown that the real part of the high-frequency impedance reads

$${\rm Re}Z^{\|} = \frac{2Z_0c^{3/2}}{(\omega g)^{3/2}} \left(\frac{2L}{\pi g}\right)^2 \left(\frac{\pi g}{l}\right)^{1/2} \tag{40}$$

where l denotes the cavity gap. This formula only applies if the collimators are positioned periodically and if, in addition,

$$\frac{2\sigma_z l}{(b-g)^2} \ll 1, \tag{41}$$

with l the distance between two adjacent collimators.

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